Error Analysis

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1 Estimating measurement errors

Lab measurement errors, often characterized by uncertainty in measured values, play a crucial role in scientific investigations and experimental analyses [1]. The inherent complexity of laboratory processes, equipment limitations, and unavoidable environmental factors contribute to uncertainties in measurements (check [2] and [3] and the Reference Material on the course page of Project 1 for more information).

1.1 Types of errors

Systematic errors, characterized by consistent and repeatable deviations from the true value, can arise from equipment limitations, calibration issues, or environmental conditions. Random errors, or statistical errors, occur unpredictably, impacting the precision of measurements. Fluctuations in experimental conditions or limitations in measurement instruments contribute to random errors. Multiple measurements and statistical analysis, such as calculating standard deviation, help mitigate random errors.

Scientists employ various methods to quantify and express these uncertainties, such as error bars, confidence intervals, and standard deviations. Understanding and acknowledging measurement uncertainties are essential for accurate data interpretation and effective communication of scientific findings.

1.2 Propagation of errors

When combining measurements in calculations, it is essential to understand how errors propagate. Consider a quantity y calculated from measured quantities $x_1, x_2, \ldots$ with a function $y = f(x_1, x_2, \ldots)$.

The propagated error Δy can be estimated using partial derivatives (check [1] and [4]). The partial derivative of y with respect to each measured quantity x_i is denoted as $\frac{\partial y}{\partial x_i}$. The total differential of y can be expressed as

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 + \ldots,$$

and the uncertainty Δy can therefore be calculated as

$$\Delta y = \sqrt{(dy)^2} = \sqrt{\left(\frac{\partial y}{\partial x_1} dx_1\right)^2 + \left(\frac{\partial y}{\partial x_2} dx_2\right)^2 + \ldots + 2\left(\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j}\right) dx_i dx_j + \ldots}$$

For random uncorrelated variables, the cross-terms are zero, simplifying the expression to

$$\Delta y = \sqrt{\sum_i \left(\frac{\partial y}{\partial x_i}\Delta x_i\right)^2} = \sqrt{\left(\frac{\partial f}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2}\Delta x_2\right)^2 + \ldots},$$

where Δx_i is the error or uncertainty in the measured quantity x_i.

For example, let Q be a calculated quantity based on measured values x, y, with uncertainties Δx, Δy. The general formula for the uncertainty ΔQ in Q is given by:

1. For the sum $Q = ax + by$ or difference $Q = ax - by$ (where a, b are constants) of quantities:

$$\Delta Q = \sqrt{(a\Delta x)^2 + (b\Delta y)^2}$$

2. For the product ($Q = ax \cdot y$) or division ($Q = a\frac{x}{y}$) (where a is a constant) of quantities:

$$\frac{\Delta Q}{Q} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

1.3 Error analysis in our radial flow experiment

In this experiment, we will be evaluating the Rossby number R_0 as a function of the azimuthal velocity, v_θ, the rotation rate of the apparatus Ω, and the distance from the center of the tank, r, such that

$$R_0 \equiv R_0(v_\theta, \Omega, r) = \frac{v_\theta}{2\Omega r}. \tag{1}$$

Each of the quantities appearing in this expression is either measured directly or calculated from other measured variables during the experiment, and has some intrinsic uncertainty that we can quantify. For example, the precision in the measured rotation speed of the tank ($\Delta\Omega$), the radius of the particle (Δr), and the estimated azimuthal velocity ($\Delta v_\theta = \Delta(r\frac{\Delta\theta}{\Delta t})$).

Hence, we can estimate the error in R_0 using the propagation of error rule, to get

$$\Delta R_0(v_\theta, \Omega, r) = \sqrt{\left(\frac{\partial R_0}{\partial v_\theta}\Delta v_\theta\right)^2 + \left(\frac{\partial R_0}{\partial \Omega}\Delta\Omega\right)^2 + \left(\frac{\partial R_0}{\partial r}\Delta r\right)^2},$$

which gives

$$\Delta R_0 = \sqrt{(\frac{\Delta v_\theta}{2\Omega r})^2 + (\frac{v_\theta \Delta\Omega}{2\Omega^2 r})^2 + (\frac{v_\theta \Delta r}{2\Omega r^2})^2}, \tag{2}$$

or a relative error of

$$\frac{\Delta R_0}{R_0} = \sqrt{(\frac{\Delta v_\theta}{v_\theta})^2 + (\frac{\Delta r}{r})^2 + (\frac{\Delta\Omega}{\Omega})^2}. \tag{3}$$

Questions

1) How are r and v_θ calculated in our experiment?
2) Can you estimate their errors given measured quantities during the experiment?
3) Do you expect any of these errors to become more significant as the particles approach the middle of the tank, spiraling quickly towards the center?
4) Which errors dominate your estimate of R_0?

1.4 Evaluating the fit between theory and experiments

From conservation of angular momentum, we were able to find a prediction for R_0 as a function of the radius r and the radius of the tank r_0,

$$R_0 = \frac{1}{2}[(\frac{r_0}{r})^2 - 1]. \tag{4}$$

How well is this prediction reproduced in our collected laboratory data?

To answer this, we can use the ”least squares” method for fitting a set of experimental data to a mathematical function (check [1] and [5]). In short, the least squares method is a mathematical approach used in regression analysis to determine the line or curve that minimizes the sum of squared differences between observed and predicted values in a given dataset.

To examine our theoretical prediction, we can re-plot our calculated Rossby number R_0, but now as a function of $(\frac{r_0}{r})^2$ instead of r (or, alternatively, using a log-log plot), and apply the least squares method by fitting the data to a straight line. Denoting $x \equiv (\frac{r_0}{r})^2$ and plotting $R_0(x)$, we can use a simple statistics software (such as MATLAB, Excel, or other) to find the best linear fit to the data (check [6] for more information),

$$y = mx + b, \tag{5}$$

with statistical estimations of the slope m and its error Δm, the intercept b and its error Δb, and the coefficient of determination value R^2 (with desirable values close to 1 and larger than at least 0.5 generally, check [7] for more information).

Questions

1) What is the slope, the intercept, and the R^2 value of the fit for your collected data points?
2) What do they tell you about the fit between our idealized theory and the experiment?
3) Can you think of systematic, random, or measurement errors that could have affected the experiment results?

References

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