Accelerations and Forces in Rotating Reference Frames

Tristan Abbott

Consider a parcel of water in a rotating tank with rotation rate Ω. The parcel has some velocity v in the rotating reference frame, and we'll measure the radial and azimuthal components of the velocity (v_r and v_θ, respectively) in a cylindrical coordinate system with origin at the center of the tank (Figure 1).

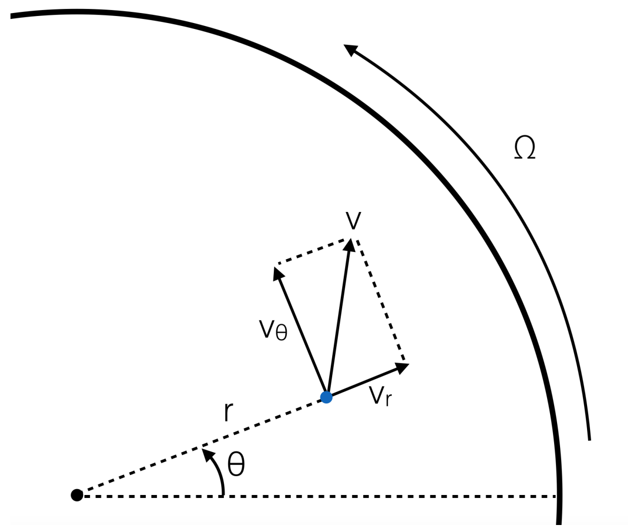

Figure 1: Position and velocity of a parcel (blue dot) in a rotating tank.

The notes for Project 1 develop an expression for a radial force balance that involves a centripetal acceleration v_θ^2/r, a Coriolis acceleration $2\Omega v_\theta$, and an inward pressure gradient force given by $g\frac{\partial h}{\partial r}$, where g is the acceleration from gravity and h is the height of the free surface (measured relative to a reference height attained in solid body rotation). This force balance, which reads

$$\frac{v_\theta^2}{r} + 2\Omega v_\theta = g\frac{\partial h}{\partial r}, \tag{1}$$

assumes that the parcel moves in a circle at constant r. This requires $v_r = 0$; otherwise, r will change. For v_r to remain zero, there cannot be any acceleration in the radial direction.

However, parcels may deviate from perfectly circular motion, and the velocity in the radial direction may not always be zero. In the radial inflow experiment, for example, parcels acquire non-zero v_r as they spiral inward. Changing the radial force balance to a radial momentum equation that includes changes in v_r is relatively simple: we just add a term for the time rate of change of v_r to the left hand side. If we adopt the sign convention that positive values indicate vectors that point toward the outside of the tank (i.e., toward large r), the radial momentum equation reads

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} - \frac{v_\theta^2}{r} - 2\Omega v_\theta = -g\frac{\partial h}{\partial r}. \tag{2}$$

Physically, this equation tells us that the net outward radial acceleration of the parcel is given by the time rate of change of the radial velocity (first term on the left-hand side), minus a centripetal acceleration (second term on the left-hand side), minus a Coriolis acceleration (third term on the left-hand side). The net acceleration must be balanced by a net force per unit mass, which is provided by a pressure gradient force (right-hand side)[1]. Figure 2 shows a schematic version of the radial momentum equation; again, the interpretation is that the net acceleration (given by the sum of the green vectors) must be equal to the pressure gradient force (the blue vector).

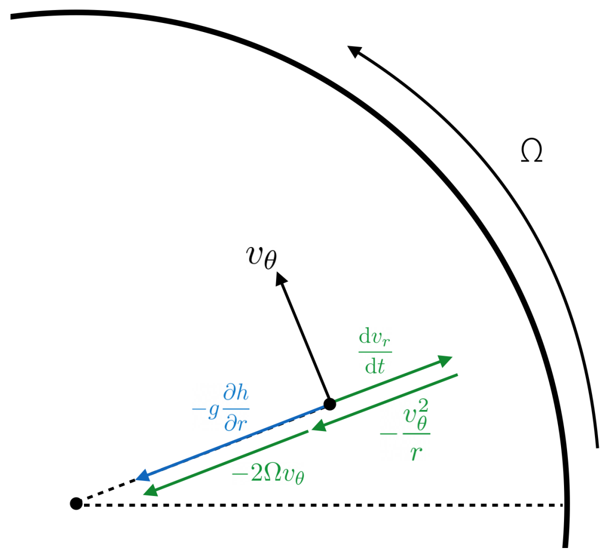

Figure 2: Radial momentum equation as expressed by Equation 2. Green vectors show components of the net acceleration, and the blue vector shows the pressure gradient force. This schematic assumes that h is higher near the edge of the tank so that the pressure gradient force is directed inward.

So why do people talk about centrifugal and Coriolis "forces"?

We live in a rotating reference frame, and the accelerations that are most obvious to us are those associated with changes to velocities *measured in the rotating frame*. We can isolate the change in radial velocity in the radial momentum equation (Equation 2) by moving the centripetal and Coriolis accelerations to the right-hand side, giving

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} = -g\frac{\partial h}{\partial r} + \frac{v_\theta^2}{r} + 2\Omega v_\theta. \tag{3}$$

When discussing this form of the radial momentum equation, the left-hand side is usually called the "acceleration", and the new terms on the right-hand side are usually called the "centrifugal force" (v_θ^2/r), and "Coriolis force" ($2\Omega v_\theta$). To be clear: the centrifugal and Coriolis forces are not forces produced by one object pushing or pulling on another; rather, they originate from accelerations associated with the rotation of the reference frame. However, as you probably know from being in a car going around a sharp corner, "centrifugal force" certainly feels real. Treating centrifugal and Coriolis terms as forces has some advantages, mainly in that it lets us think about relationships between forces and changes in measured velocities in a way that feels similar to physics in familiar non-rotating reference frames. The main difficulty is that, while we have good intuition for centrifugal forces, we need to develop it for Coriolis forces. Resources for visualizing motion

[1]Strictly speaking, the full pressure gradient force is $g\frac{\partial H}{\partial r}$, where H is the height of the free surface above the bottom of the tank. Because part of the full pressure gradient force provides a fixed centripetal acceleration set by the rotation rate of the tank, though, the "dynamics" (meaning, the time evolution of v_r and v_θ) are only affected by the remaining portion of the pressure gradient force, which is given by $g\frac{\partial h}{\partial r}$.

under the influence of Coriolis forces (cmpo5.mit.edu/8.01) and visualizing the relationship between velocity, rotation rate, and Coriolis force (thabbott.github.io/d3/coriolis.html) may be helpful for building this intuition.

Physically, Equation 3 tells us that the time rate of change of the radial velocity (left-hand side) is set by the combination of a pressure gradient force (first right-hand side term), a centrifugal force (second right-hand side term), and a Coriolis force (third right-hand side term). Figure 3 shows a schematic version of this equation. Note the similarity with Figure 2: the two schematics are identical apart from changes in the color and direction of two vectors.

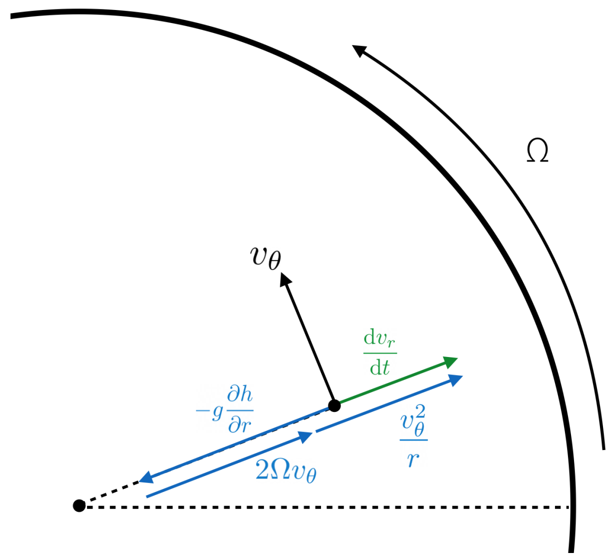

Figure 3: Radial momentum equation as expressed by Equation 3. The green vector shows the time rate of change of the radial velocity, and the blue vectors show components of the net force, including centrifugal and Coriolis forces. This schematic assumes that h is higher near the edge of the tank so that the pressure gradient force is directed inward.

As an exercise using this form of the radial momentum equation, consider a situation where the pressure gradient force is directed inward, the azimuthal velocity is positive, and parcels follow circular trajectories at constant r. Circular flow requires zero radial velocity ($v_r = 0$), and the radial momentum equation becomes

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} = -g\frac{\partial h}{\partial r} + \frac{v_\theta^2}{r} + 2\Omega v_\theta = 0. \tag{4}$$

Now, imagine that one of two things happens:

1. The inward pressure gradient force becomes stronger. This corresponds to $-g\frac{\partial h}{\partial r}$ becoming more negative. The radial momentum equation will change so that

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} = -g\frac{\partial h}{\partial r} + \frac{v_\theta^2}{r} + 2\Omega v_\theta < 0, \tag{5}$$

   v_r will become negative, and parcels will begin to move inward toward the center of the tank.

2. The azimuthal velocity increases. This will produce stronger outward centrifugal and Coriolis forces because both v_θ^2/r and $2\Omega v_\theta$ will become larger. The radial momentum equation will

change so that

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} = -g\frac{\partial h}{\partial r} + \frac{v_\theta^2}{r} + 2\Omega v_\theta > 0, \tag{6}$$

v_r will become positive, and parcels will begin to move outward toward the edge of the tank.

How does this help us think about the radial inflow experiment?

Imagine starting with the tank in solid-body rotation, with $v_\theta = v_r = 0$. We open the drain at the center of the tank and start pumping water into the tank at its outer edge, causing the free surface to fall in the center of the tank and rise at the edge. This makes $\frac{\partial h}{\partial r}$ larger, increasing the inward pressure gradient force, and parcels begin to move inward toward the center of the tank. As parcels move inward, however, they conserve angular momentum, and their azimuthal velocity increases. This produces stronger outward centrifugal and Coriolis forces. As centrifugal and Coriolis forces increase, they reduce the inward acceleration of water parcels. At high rotation rates, where Ω is large and angular momentum conservation produces large azimuthal velocities, this makes it hard for parcels to accelerate rapidly toward the center of the tank, and they circulate around the tank many times before eventually reaching the drain. This is what we observe in the radial inflow experiment: at low rotation rates, particles move in relatively straight lines toward the middle of the tank, while at high rotation rates, particles spiral inwards relatively slowly and complete many circuits around the tank before reaching the drain.