2 Fluid laboratory: Radial inflow experiment

2.1 Experimental procedure

The laboratory experiment — known as 'radial inflow' or 'balanced vortex' — is a more controlled, perpetual motion version of the 'hole in a bucket' experiment just described. We rotate a cylinder about its vertical axis, as shown in Fig.4; the cylinder has a circular drain hole in the center of its bottom. Water enters at a constant rate through a diffuser on its outer wall and exits through the drain. The water is then pumped back around enabling a steady state to be achieved. The system is viewed from above by a camera which is rotating with the turntable — i.e. in the rotating frame of reference.

When the apparatus is rotating the water acquires a pronounced swirling motion as it exits the tank: fluid parcels *spiral* inward as sketched in Fig.5 (rhs). Even at modest rotation rates of $\Omega = 1$ radian per second (corresponding to a rotation period of around 6 seconds)[1], the effect of rotation is marked and parcels complete many circuits before finally exiting through the drain hole. In the presence of rotation the free surface becomes markedly curved, high at the periphery and plunging downwards toward the hole in the center. When the apparatus is not rotating, particle trajectories are less distinctive but are generally directed radially inward (as sketched in Fig.5, lhs) and the free surface is observed to be rather flat.

2.2 Measurements and experimental procedure

The main objective of our experiment is to measure, and interpret in terms of mechanics and angular momentum principles (see theory below), the flow field and its dependence on rotation rate. We will also think about the relation between the flow field and the pressure field, given by the height of the free surface. This can exhibit pronounced curvature, diving down toward the hole in the middle, particularly in the case of higher rotation rates.

We carry out three experiments, one at low rotation rates (1rpm), medium (6rpm) and high (rpm) — don't try and break the apparatus by making it fly off the table!

In each experiment we:

1. record data from the overhead camera, rotating with the turntable, showing the position of particles as a function of time using the particle tracker provided,

2. plot trajectories of chosen particles (using Excel or Matlab, or your program of choice),

[1] Note that if Ω is the rate of rotation of the tank in radians per second, then the period of rotation is $\tau_{\text{tank}} = \frac{2\pi}{\Omega}$. Thus if $\tau_{\text{tank}} = 2\pi s$, then $\Omega = 1$.

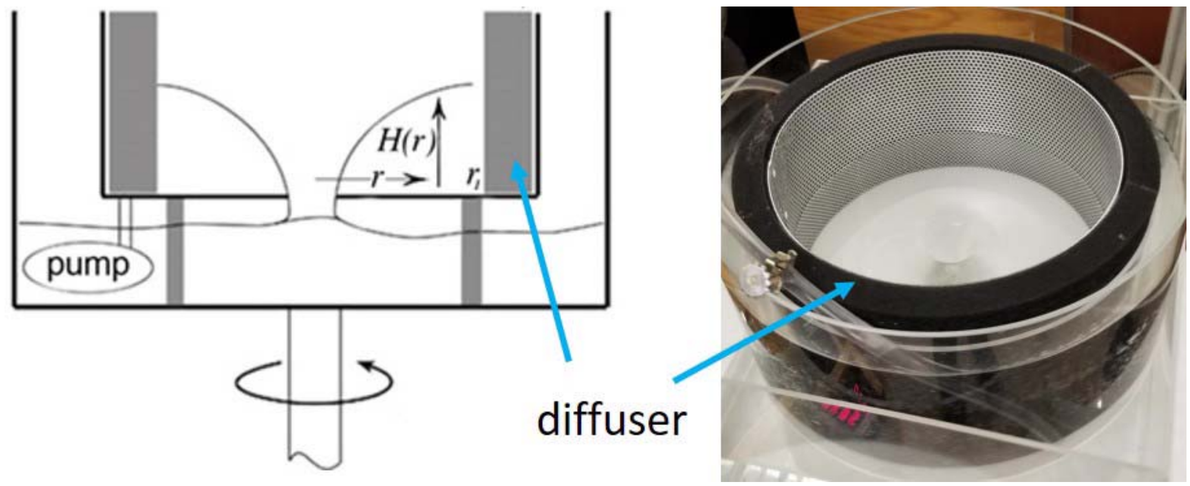

Figure 4: The radial inflow apparatus. A cylindrical diffuser is placed in a larger tank and used to produce an axially symmetric, inward flow of water toward a drain hole at the center. Below the tank there is a large catch basin, partially filled with water and containing a submersible pump whose purpose is to return water to the diffuser in the upper tank. The whole apparatus is then placed on a turntable and rotated in an anticlockwise direction. The path of the fluid is visulised by tracking paper dots on the free surface.

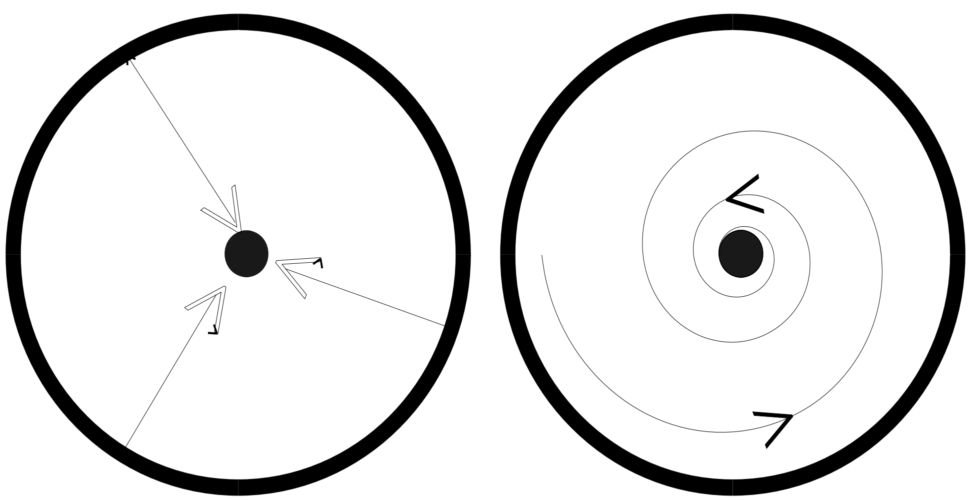

Figure 5: Idealized flow patterns (left) in the absence of rotation and (right) when the apparatus is rotating in an anticlockwise direction.

Figure 6: Trajectory of a particle in the rotating frame as it circulates around in the swirling vortex.

3. compute particle velocities as a function of radius,

4. plot the Rossby number, defined in Eq.(12) below, as a function of radius,

5. interpret in terms of the prediction from theory, Eq.(14), presented below.

Measuring position and velocity of particles. We record a sequence of images using the overhead camera and deploy particle tracking software on the laboratory computers to track chosen particles. An example is shown in Fig.6. This returns the coordinates of individual particles as a function of time (frame number). From this we can compute both the azimuthal (v_θ) and radial (v_r) velocity of the particles in the rotating frame of reference, using a cylindrical coordinate system — see Fig.7.

From this data we can test whether the azimuthal speed of the dots, $v_\theta(r)$, is consistent with angular momentum conservation, Eq.(4) below. We can also compute how the Rossby number, given by Eq.(12), varies across the vortex. This can be compared to the theoretical prediction, Eq.(14), that would pertain if the particles did indeed conserve angular momentum. This radial distribution of Rossby number will be a quantity that we compute from meteorological observations in the 'atmospheric data' component of our project.

Tips on how to use the particle tracker and how one converts from Cartesian to cylindrical

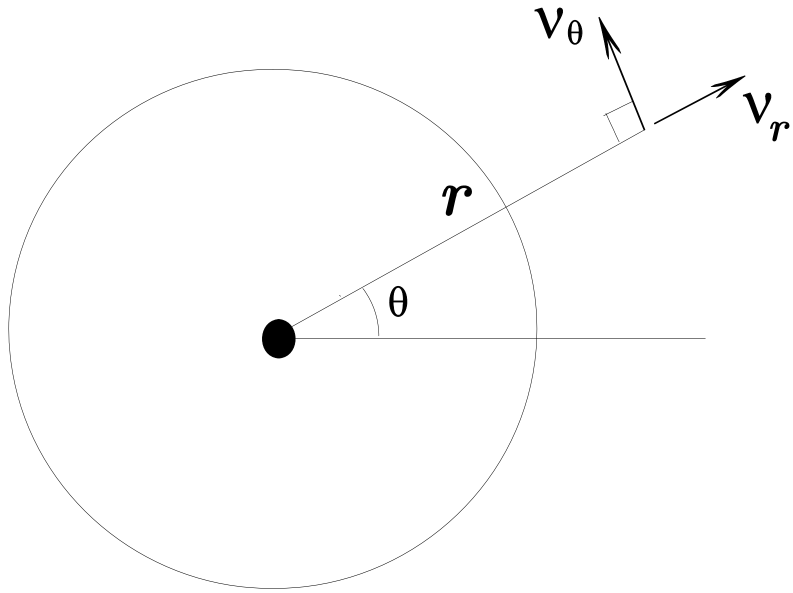

Figure 7: The velocity of a fluid parcel viewed in the rotating frame of reference: $v_{rot} = (v_r, v_\theta)$ in radial and azimuthal directions, (r, θ) coordinates.

coordinates (so as to compute, for example, $v_\theta(r)$) can be found on our course website.

2.3 Theory of a balanced vortex

Frames of reference. If V_θ is the azimuthal velocity in the absolute frame (the frame of the laboratory) and v_θ is the azimuthal speed *relative* to the tank (measured using the camera co-rotating with the apparatus) then (see Fig.7):

$$V_\theta = v_\theta + \Omega r, \tag{2}$$

where Ω is the rate of rotation of the tank in radians per second. Note that Ωr is the azimuthal speed of a particle fixed to the tank at radius r from the axis of rotation.

Angular momentum. Fluid entering the tank at the outer wall will have angular momentum because the apparatus is rotating. At r_1, the radius of the diffuser, fluid has azimuthal velocity Ωr_1 and hence angular momentum Ωr_1^2. As parcels of fluid flow inwards axi-symmetrically toward the central hole, they might be expected to conserve this angular momentum (provided that they are not rubbing against the bottom or the side, so that we may ignore friction). Conservation of angular momentum states that:

$$V_\theta \, r = \text{constant} = \Omega r_1^2. \tag{3}$$

Here V_θ is the azimuthal velocity in the laboratory (inertial) frame given by Eq.(2). Combining Eqs.(3) and (2) we find that the azimuthal speed in the rotating frame is:

$$v_\theta = \Omega \frac{(r_1^2 - r^2)}{r} \ . \tag{4}$$

Thus $v_\theta = 0$ at $r = r_1$ and v_θ increases as r decreases. We can readily test this prediction against the observations of trajectories obtained in the laboratory using the particle tracker.

We now consider the balance of forces in the vertical and radial directions, expressed first in terms of the absolute velocity V_θ and then in terms of the relative velocity v_θ.

Vertical force balance. We suppose that the pressure at any depth in the fluid is set by the weight of water above that level[2]:

$$\underbrace{p}_{\text{force/unit area}} = -\underbrace{\rho z}_{\text{mass/unit area}} \times \underbrace{g}_{\text{gravity}} + \text{const}$$

where ρ is the (constant) density, g is the acceleration due to gravity and z is a vertical coordinate increasing upwards (from $z = 0$ at the base of the tank). Note the minus sign ensures that pressure decreases on going upwards. At the free surface, at height $H(r)$ from the bottom, we suppose that the pressure vanishes (actually $p =$ atmospheric pressure at the surface, which can be taken as zero). Thus, we have:

$$p(r, z) = \rho g \left(H(r) - z\right) . \tag{5}$$

At the bottom $p = \rho g H$ and p decreases linearly to zero at the free surface where $z = H$.

Radial force balance in the non-rotating frame. If the pitch of the spiral traced out by fluid particles is tight (*i.e.* in the limit that $\frac{v_r}{v_\theta} << 1$, appropriate when Ω is sufficiently large) then the centrifugal force directed radially outwards acting on a particle of fluid is balanced by the pressure gradient force directed inwards associated with the tilt of the free surface. This radial force balance can be written in the non-rotating frame thus:

$$\underbrace{\frac{V_\theta^2}{r}}_{\text{Centrifugal acc}^{\text{n}}} = \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial r}}_{\text{pressure gradient}} \ .$$

[2]The differential form of this relation is:

$$\frac{\partial p}{\partial z} + \rho g = 0$$

which is known as 'hydrostatic balance', the pressure distribution that pertains in a resting fluid.

Using Eq.(5), the radial pressure gradient force in the above can be directly related to the gradient of the free surface enabling the force balance to be written:

$$\frac{V_\theta^2}{r} = g\frac{\partial H}{\partial r}. \tag{6}$$

Radial force balance in the rotating frame. Using Eq.(2), we can express the centrifugal acceleration in Eq.(6) in terms of velocities in the rotating frame thus:

$$\frac{V_\theta^2}{r} = \frac{(v_\theta + \Omega r)^2}{r} = \frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r \tag{7}$$

Hence our radial momentum balance becomes

$$\frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r = g\frac{\partial H}{\partial r}. \tag{8}$$

Solid body rotation state. Note that if the water in the tank is at rest in the rotating frame of reference, i.e. $v_\theta = 0$ in Eq.(8) (this in known as solid body rotation) then

$$\frac{\partial H_{\text{solid body}}}{\partial r} = \frac{\Omega^2 r}{g} \text{ and so } H_{\text{solid body}} = H_{r=0} + \frac{\Omega^2 r^2}{2g}, \tag{9}$$

where $H_{r=0}$ is the height of the water at $r = 0$ in the solid body rotation state. Note that the free surface in the solid body rotation state has a parabolic form. The surface is tilted in the steady state because the inward acceleration due to the tilt of the free surface ($g\frac{\partial H}{\partial r}$) is exactly balanced by the centrifugal acceleration directed outward $\Omega^2 r$. See Fig.8

The Rossby number and geostrophic balance. Eq.(8) can be simplified by writing $\Omega^2 r = \frac{\partial}{\partial r}\left(\frac{\Omega^2 r^2}{2}\right)$ and defining a quantity h:

$$h = H - H_{\text{solid body}} = H - \frac{\Omega^2 r^2}{2g}, \tag{10}$$

the height of the free surface measured relative to that of the reference parabolic surface, using Eq.(9) — see Fig.9. Then Eq.(8) can be written in term of h thus:

$$\frac{v_\theta^2}{r} = g\frac{\partial h}{\partial r} \underbrace{-2\Omega v_\theta}_{\text{Coriolis acc}^{\text{n}}} : \quad \text{gradient wind balance} \tag{11}$$

This balance of forces is known as 'gradient wind balance'.

Eq.(6) and Eq.(11) are completely equivalent statements of the balance of forces. The distinction between them is that the former is expressed in terms of V_θ, the latter in terms

Figure 8: The free surface of a fluid in solid body rotation takes up a parabolic shape, as can be seen in this large square tank of water rotating at 20rpm.

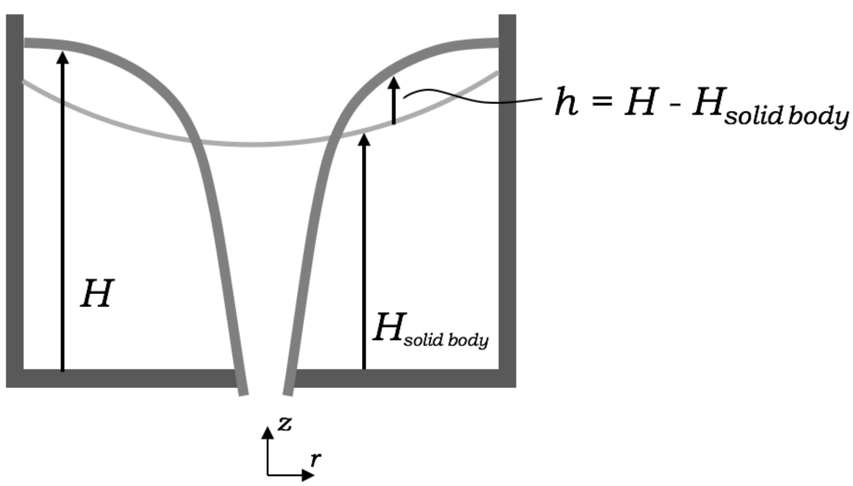

Figure 9: Height of the free surface, h, relative to solid body: $h = H - H_{\text{solid body}}$. H is the actual height of the free surface

of v_θ. Note that Eq.(11) has the same form as Eq.(6) except:

1. an extra term, $-2\Omega v_\theta$, appears on the rhs of Eq.(11) — this is called the 'Coriolis acceleration'. It has appeared because we have chosen to express our force balance in terms of *relative*, rather than absolute velocities.

2. the pressure gradient force is expressed in term of h (the height measured relative to the reference parabolic surface) rather than H.

Let us compare the magnitude of the centrifugal acceleration, $\frac{v_\theta^2}{r}$, and the Coriolis acceleration, $2\Omega v_\theta$, terms in Eq.(11). Their ratio is the 'Rossby number'[3]:

$$R_o = \frac{|v_\theta|}{2\Omega r} \tag{12}$$

If $R_o << 1$, the $\frac{v_\theta^2}{r}$ term can be neglected in (11). In this limit, Coriolis and pressure gradient terms balance one another.

$$g\frac{\partial h}{\partial r} - 2\Omega v_\theta = 0: \quad \text{geostrophic balance} \tag{13}$$

Equation (13) is a simple form of the 'geostrophic equation' relating velocities in the rotating frame to the horizontal pressure gradient in the limit of small R_o.

How large is R_o in our experiment? We can estimate its size by computing v_θ based on angular momentum conservation. Using the angular momentum conserving prediction (4) in (12) we find, for this profile,

$$R_o = \frac{1}{2}\left(\left(\frac{r_1}{r}\right)^2 - 1\right). \tag{14}$$

Thus $R_o = 1$ at $r = r_1/\sqrt{3}$; $R_o < 1$ if $r > r_1/\sqrt{3}$ (the region of geostrophic balance) and so, in the outer regions of the flow, the inward radial pressure gradient is balanced by outward Coriolis forces (small R_o): the flow is in geostrophic balance here. In the inner regions, $r < r_1/\sqrt{3}$, $R_o > 1$. But as parcels spiral into the drain they pass through a region where R_o becomes increasingly large and v_θ^2/r in Eq.(11) becomes a dominant term. In the limiting

[3]It is interesting to compare the Rossby number defined here with the ratio of timescales $R_{timescales}$ written down in Eq.(1) and used in our initial exploration of our laboratory vortex:

$$R_{timescales} = \frac{2\pi/\Omega}{2\pi r/v_\theta} = \frac{v_\theta}{\Omega r} = 2 \times R_o.$$

Thus $R_{timescales}$ is exactly twice the Rossby number. The two non-dimensional numbers are essentially equivalent to one another, apart from the numerical factor of 2.

case $R_o \gg 1$, the centrifugal term dominates the Coriolis term in Eq.(11), and we have what is known as "cyclostrophic balance:"

$$\frac{v_\theta^2}{r} = g\frac{\partial h}{\partial r} : \quad \text{cyclostrophic balance} \tag{15}$$

Non-dimensional expressions. When making connections between the real world phenomena and laboratory abstractions (e.g. the flow in a hurricane and our laboratory vortex), it is useful to make use of non-dimensional numbers such as the Rossby number. If we non-dimensionalize distance by r_1 and velocity by Ωr_1 (the speed that a particle has as it exits from the diffuser) then we may define non-dimensional distance and speed thus:

$$\widetilde{r} = \frac{r}{r_1}, \ \widetilde{v_\theta} = \frac{v_\theta}{\Omega r_1}.$$

Then Eqs.(4 & 14) can be written:

$$\widetilde{v_\theta} = \frac{(1 - \widetilde{r}^2)}{\widetilde{r}}; \quad R_o = \frac{1}{2}\left(\left(\frac{1}{\widetilde{r}}\right)^2 - 1\right) \tag{16}$$

Thus $\widetilde{v_\theta}$ and R_o only depend on $\widetilde{r}$ and not on Ω! Is this true of the data you analyze in the radial flow experiment?

We will see that the range of Rossby numbers in our laboratory vortex — from large in the center to small on the periphery — are very similar to those found in Hurricanes and strong vortices on Earth, suggesting that the dynamical balances in the latter are the same as captured in our laboratory.