1   The relationship between pressure and density: hydrostatic balance

Notes taken from 12.003.

Let us imagine that the atmospheric T profile is as observed, for example, in Fig.(1). What is the implied vertical distribution of p and ρ? If the atmosphere were at rest – static – then pressure at any level would depend on the weight – and therefore density profile – of the fluid above that level. This balance, which we now discuss in detail, is called *hydrostatic balance*.
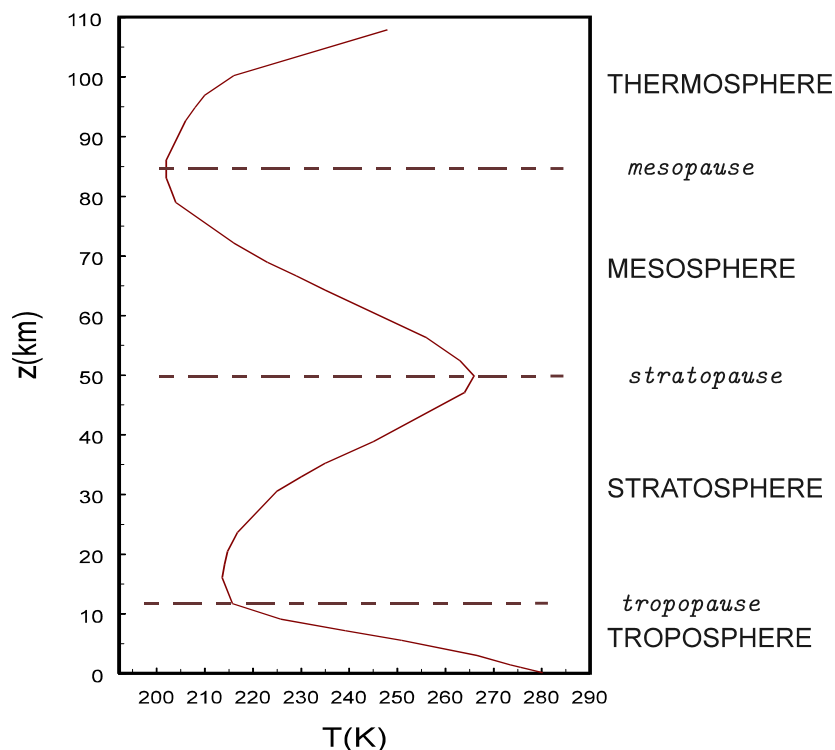
Figure 1: Vertical temperature profile for the 'US' standard atmosphere at $40°N$ in December.

Consider Fig.2. It depicts a vertical column of air of horizontal cross-sectional area δA and depth δz. Pressure $p(z)$ and density $\rho(z)$ of the air are both expected to be functions of z (they may be functions of x, y, and t also). If the pressure at the top of the cylinder is $p_T = p(z)$, then that at the bottom is

$$
\begin{aligned}
p_B &= p(z - \delta z) \\
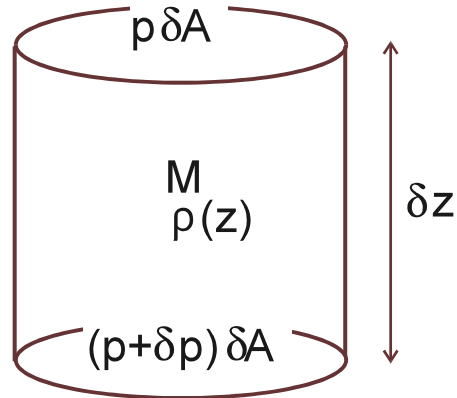&= p(z) + \delta p,
\end{aligned}
$$

Figure 2: Hydrostatic balance

where, assuming δz to be small,

$$\delta p = -\frac{\partial p}{\partial z}\delta z \ . \tag{1}$$

Now, the mass of the cylinder is

$$M = \rho \, \delta A \, \delta z \ .$$

If the cylinder of air is not accelerating, it must be subjected to zero net force. The vertical forces (upward being positive) are:

i) gravitational force $F_g = -gM = -g\rho \, \delta A \, \delta z$,

ii) pressure force acting at the top face, $F_T = -p \, \delta A$, and

iii) pressure force acting at the bottom face, $F_B = (p + \delta p) \, \delta A$.

Setting the net force $F_g + F_T + F_B$ to zero gives $\delta p = g\rho\delta z$, and using Eq.(1)[1] we obtain:

$$\frac{\partial p}{\partial z} + g\rho = 0 \tag{2}$$

Eq.(2) is the equation of **hydrostatic balance.** It describes how pressure must decrease with height in order to support the mass of the overlying atmosphere. Note that, since p must vanish as $z \to \infty$ - the atmosphere fades away - we can integrate Eq.(2) from z to ∞

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[1]A note on notation for the partial derivatives. In general, all variables are functions of the three spatial coordinates x, y, and z, and of time t. So when a partial derivative is written without any other qualification, it means holding all other coordinates constant. In Eq. (2), therefore, $\partial/\partial z$ implies "at constant x, y, and t" – see appendix.

to give

$$p(z) = g \int_z^\infty \rho \, dz$$

and that $\int_z^\infty \rho \, dz$ is just the mass per unit area of the atmospheric column above z.) The only important assumption we made in the derivation of Eq.(2) was the neglect of any vertical acceleration of the cylinder (in which case, the net force need not be zero). This is an excellent approximation under almost all circumstances in both the atmosphere and ocean. It can become suspect, however, in very vigorous small-scale systems in both the atmosphere and ocean (*e.g.*, tornados, violent thunderstorms and deep, polar convection in the ocean).

Note that Eq.(2) does not tell us what $p(z)$ is, since we do not know *a priori* what $\rho(z)$ is. In order to determine $p(z)$, we have to invoke an equation of state to tell us the connection between ρ and p.