Geostrophic balance

John Marshall, Alan Plumb and Lodovica Illari

March 4, 2003

Abstract

We describe the theory of Geostrophic Balance, derive key equations and discuss associated physical balances.[1]

1 Geostrophic balance

If the flow is such that the Rossby number is small — $R_o \ll 1$ — where

$$R_o = \frac{U}{fL} \tag{1}$$

(here U is a typical horizontal current speed, f is the Coriolis parameter and L is a typical horizontal scale over which U varies), then the Coriolis force is balanced by the pressure gradient force in the horizontal component of the momentum equation, which reduces to:

$$f\widehat{\mathbf{z}} \times \mathbf{u} + \frac{1}{\rho}\nabla p = 0 \tag{2}$$

where $\widehat{\mathbf{z}}$ is a unit vector in the vertical direction.

In situations where (as is always the case) Eq.(2) is only an approximate balance, the velocity $\mathbf{u}$ defined by (2) — involving only the horizontal components of $\mathbf{u}$ — is known as the *geostrophic wind (or current)*. Rewriting Eq.(2), we can define the geostrophic wind (since $\widehat{\mathbf{z}} \times \widehat{\mathbf{z}} \times \mathbf{u} = -\mathbf{u}$) as

$$\mathbf{u}_g = \frac{1}{\rho f}\widehat{\mathbf{z}} \times \nabla p \ , \tag{3}$$

or, writing out its Cartesian components,

$$\begin{aligned} u_g &= -\frac{1}{\rho f}\frac{\partial p}{\partial y} \ ; \\ v_g &= \frac{1}{\rho f}\frac{\partial p}{\partial x}. \end{aligned} \tag{4}$$

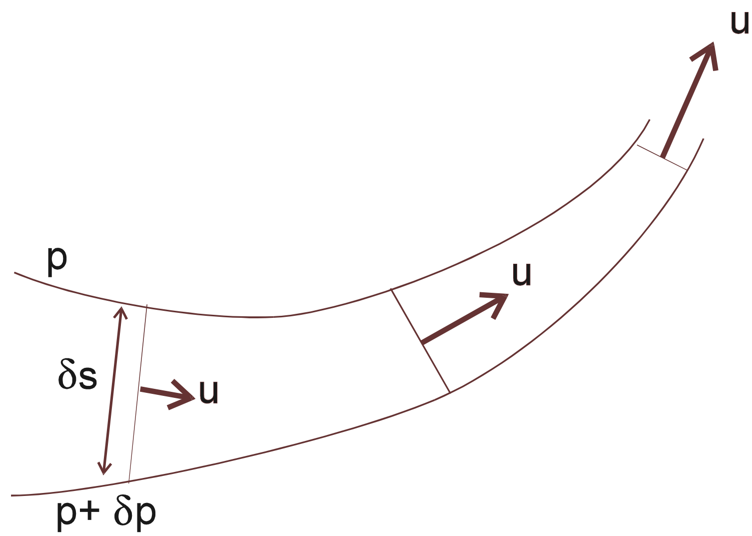

Figure 1: Schematic of two isobars on a horizontal surface. The magnitude of u increases as the siobars become closer together.

We see from Eq.(2) that the geostrophic flow is normal to the pressure gradient: i.e. along isobars (lines of constant pressure) and its speed is proportional to the pressure gradient. Consider Fig.1, the curved lines show two isobars on which pressure has the constant values p and $p + \delta p$. Their separation is δs. From Eq.(3), the flow speed is

$$|\mathbf{u}_g| = \frac{1}{\rho f} |\nabla p| = \frac{1}{\rho f} \frac{\delta p}{\delta s} .$$

Since δp is constant along the flow, $|\mathbf{u}_g| \propto (\delta s)^{-1}$: the flow is strongest where the isobars are closest together. Since the geostrophic flow cannot cross the isobars, the latter act like banks of a river, causing the flow to speed up where the river is narrow and slow down where it is wide.

As shown in Fig.2, the flow is (in the northern hemisphere) anticlockwise (cyclonic) around a low pressure center, and clockwise (anticyclonic) around a center of high pressure. ("Cyclonic" means in the same sense as the vertical component of the Earth's rotation, and "anticyclonic" the opposite. So, in the southern hemisphere where $f < 0$, the flow is clockwise, but still cyclonic, around a low pressure center.) This rule is summarized in

[1] Notes to accompany 12.307: Weather and Climate Laboratory. For a more detailed description see notes on 12.003 web page here: http://paoc.mit.edu/labweb/notes.htm

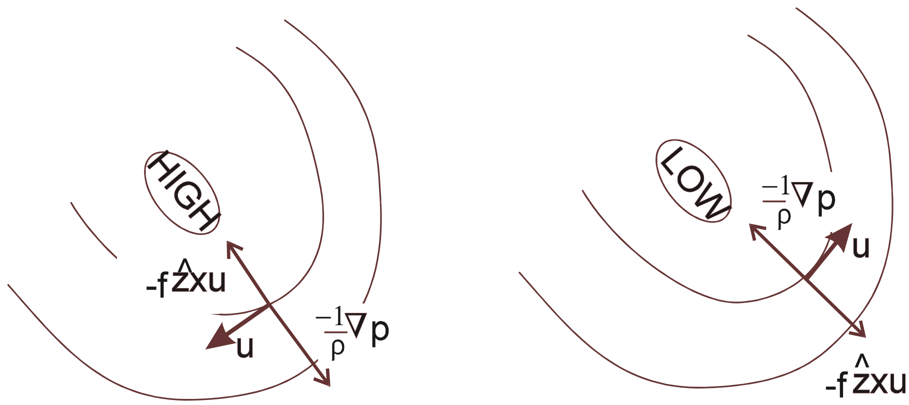

Figure 2: Geostrophic flow around (left) a low pressure center and (right) a high pressure center. (Northern hemisphere case, $f > 0$.) The effect of Coriolis delecting flow 'to the right' is balanced by the horizontal component of the pressure gradient force, $-\frac{1}{\rho}\nabla p$, directed from high to low pressure.

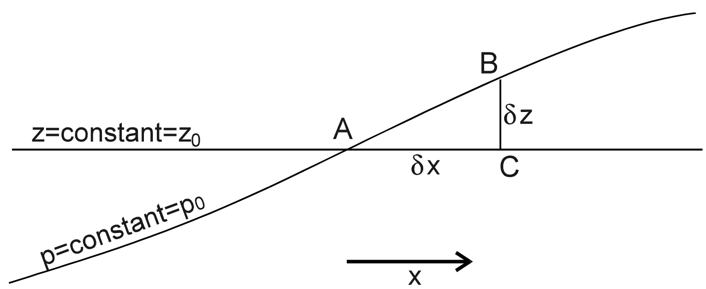

Figure 3:

Buys-Ballot's law:

If you stand with your back to the wind in the northern
hemisphere, low pressure is on your left.

We now consider pressure coordinate versions of the geostrophic and continuity equations which allow some simplifications (with regard to density variations) to be made.

1.1 The geostrophic wind in pressure coordinates

In order to apply the geostrophic equations to atmospheric observations and particularly upper air analyses (see below), we need to express them in terms of height gradients on a pressure surface, rather than, as in Eq.(4), of pressure gradients at constant height. Consider Fig.3. The figure depicts a surface of constant height z_0, and one of constant height p_0, which intersect at A, where of course pressure is $p_A = p_0$ and height is $z_A = z_0$. At constant height, the gradient of pressure in the x-direction is

$$\left(\frac{\partial p}{\partial x}\right)_z = \frac{p_C - p_0}{\delta x}$$

where δx is the (small) distance between C and A. Now, the gradient of height along the constant pressure surface is

$$\left(\frac{\partial z}{\partial x}\right)_p = \frac{z_B - z_0}{\delta x} .$$

Since $z_C = z_0$, and $p_B = p_0$, we can use the hydrostatic balance equation Eq.(**??**) to write

$$\frac{p_C - p_0}{z_B - z_0} = \frac{p_C - p_B}{z_B - z_C} = -\frac{\partial p}{\partial z} = +g\rho$$

Therefore (and invoking a similar result in the y-direction), it follows that

$$\begin{aligned} \left(\frac{\partial p}{\partial x}\right)_z &= g\rho\left(\frac{\partial z}{\partial x}\right)_p ; \\ \left(\frac{\partial p}{\partial y}\right)_z &= g\rho\left(\frac{\partial z}{\partial y}\right)_p . \end{aligned}$$

Therefore Eq.(3) becomes, in pressure coordinates,

$$\mathbf{u}_g = \frac{g}{f}\widehat{\mathbf{z}_p} \times \nabla_p z , \tag{5}$$

where $\widehat{\mathbf{z}_p}$ is the upward unit vector in pressure coordinates and ∇_p denotes the gradient operator in pressure coordinates. In component form,

$$(u_g, v_g) = \left(-\frac{g}{f}\frac{\partial z}{\partial y}, \frac{g}{f}\frac{\partial z}{\partial x}\right) . \tag{6}$$

Note that, like p contours on surfaces of constant z, z contours on constant p are streamlines of the geostrophic flow. Note that, in pressure coordinates, the geostrophic wind is, as before, nondivergent if f is taken as constant:

$$\nabla \cdot \mathbf{u}_g = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .$$

Let's now look at some synoptic charts to see some of these ideas in action.

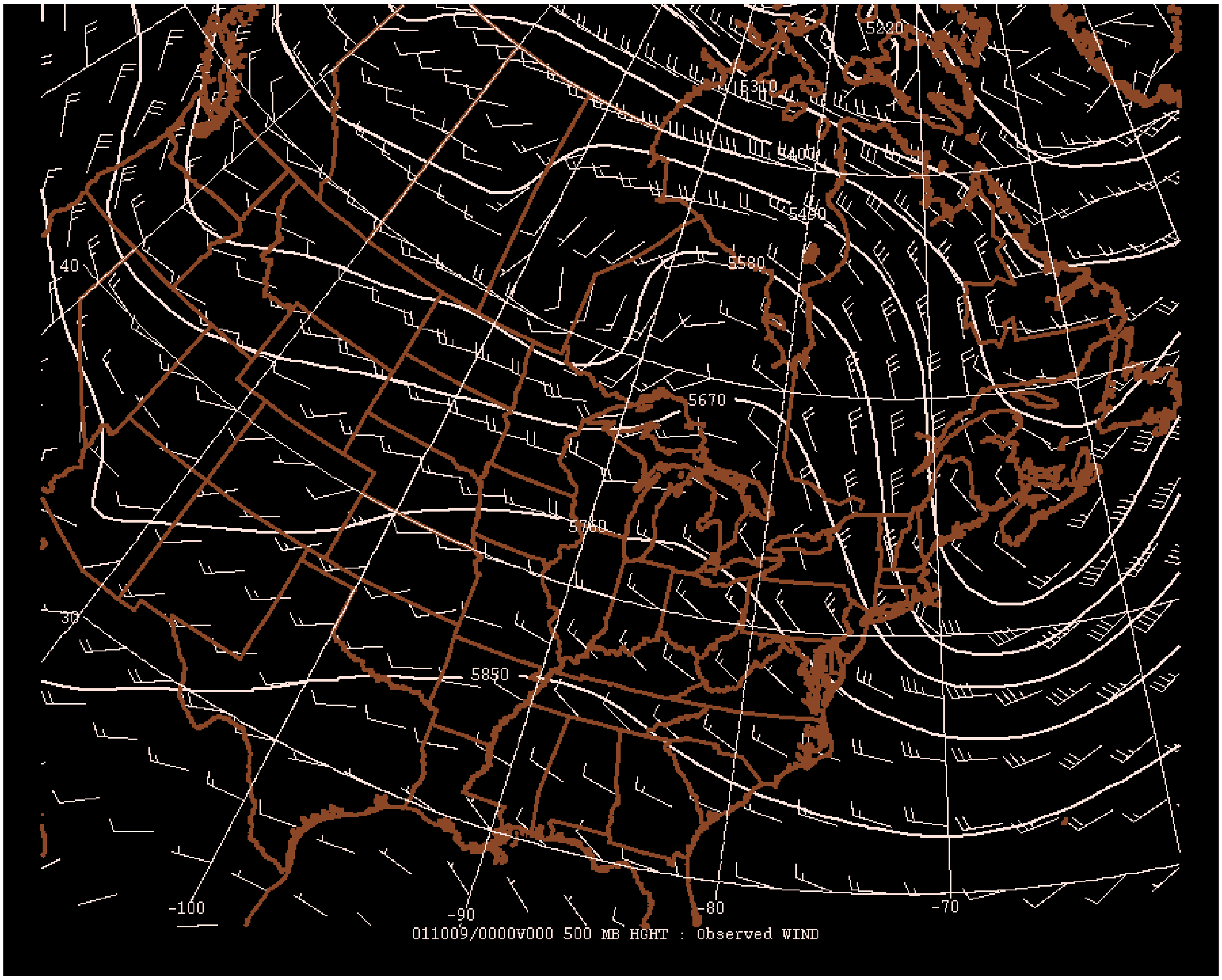

Figure 4: 500mb wind and geopotential height field on October 9th 2001. The wind blows away from the quiver: one full quiver denotes a speed of $5ms^{-1}$, one half-quiver a speed of $2.5ms^{-1}$. The geopotential height is in meters.

1.2 Highs and Lows; synoptic charts

Fig.4 shows the height of the 500mb surface (in geopotential metres, contoured) plotted with the observed wind vector (one full quiver represents a wind speed of 5 ms^{-1}). Note how the wind blows along the isobars and is strongest the closer the isobars are together - see the schematic diagram in Fig.2. At this level, away from frictional effects at the ground, the wind is close to geostrophic.

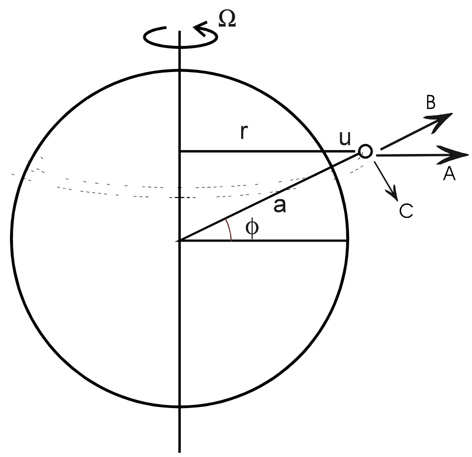

Figure 5:

1.3 Visualizing geostrophic balance on the sphere

To get a feel for geostrophic balance on the sphere it is very useful to consider a ring of air moving eastward at speed u relative to the underlying rotating earth.

There is a centrifugal acceleration directed outwards perpendicular to the earth's axis of rotation, the vector A sketched in fig(5):

$$\frac{V^2}{r} = \frac{(u + \Omega r)^2}{r} = \Omega^2 r + 2\Omega u + \frac{u^2}{r} \tag{7}$$

Here V is the 'absolute' velocity the fluid has viewed from an observer fixed in space looking back at the earth. Let's now consider the terms in turn:

- $\Omega^2 r$ - this is the centrifugal acceleration acting on a particle fixed to the earth. As discussed above, this acceleration is included in the gravity which is usually measured and is the reason that the earth is not a perfect sphere.

- $2\Omega u + \frac{u^2}{r}$ - the additional centrifugal acceleration due to motion relative to the earth. Note that if $\frac{u}{\Omega r} << 1$, we may neglect the term in u^2. For the earth $R_o = \frac{u}{\Omega r} \sim 0.02$ and so the $2\Omega u$ term dominates. It is directed outward perpendicular to the axis of rotation and can be resolved: perpendicular to the earth's surface - vector B in the diagram - and parallel to the earth's surface - vector C in the diagram.

Component B changes the weight of the ring slightly - it is very small compared to g, the acceleration due to gravity, and so unimportant.

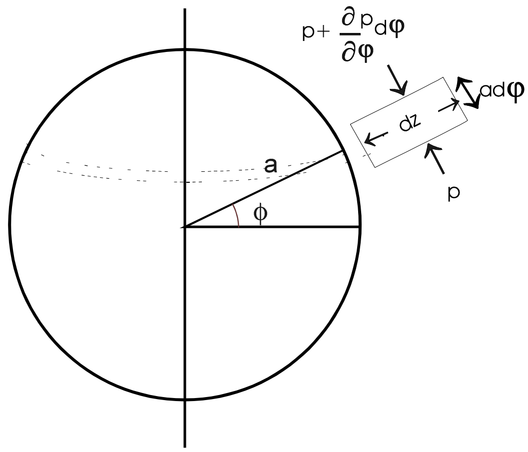

Figure 6:

Component C, parallel to the earth's surface, is the Coriolis acceleration:

$$2\Omega \sin\varphi \times u$$

So there is a centrifugal force directed toward the equator because of the motion of the ring of air relative to the earth. It is this force that balances the pressure gradient force associated with the sloping isobaric surfaces induced by the pole-equator temperature gradient.

Let's postulate a balance between the Coriolis force and the pressure-gradient force directed from equator to pole associated with the tilted isobaric surfaces - see Fig.6.

$$\underbrace{\rho a d\varphi dz}_{mass} \times \underbrace{2\Omega \sin\varphi u}_{acceleration} = \underbrace{-\frac{\partial p}{\partial \varphi} d\varphi dz}_{p\_grad}$$

Introducing a coordinate y which points northwards on the earth's surface, $dy = ad\varphi$, the above reduces to:

$$fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \tag{8}$$

where $f = 2\Omega \sin\varphi$ is the Coriolis parameter.

This is just one component of the geostrophic relation, Eq.(4), between pressure gradient forces and Coriolis forces.