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Subgeostrophic flow: the effect of friction

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1 Subgeostrophic flow: the Ekman layer

The horizontal component of the momentum equation in the presence of friction, \mathcal{F} , is, in the limit of small Rossby number:

$$f\hat{\mathbf{z}} \times \mathbf{u} + \frac{1}{\rho}\nabla p = \mathcal{F} \quad (1)$$

The horizontal flow is not now geostrophic. Rather, the horizontal flow \mathbf{u}_h is

$$\mathbf{u}_h = \mathbf{u}_g + \mathbf{u}_{ag} \quad (2)$$

where \mathbf{u}_{ag} is the *ageostrophic* current, the departure of the actual horizontal flow from its geostrophic value \mathbf{u}_g given by:

$$\mathbf{u}_g = \frac{1}{\rho f}\hat{\mathbf{z}} \times \nabla p.$$

Using Eq.(1), Eq.(2) and the geostrophic relation, we see that:

$$f\hat{\mathbf{z}} \times \mathbf{u}_{ag} = \mathcal{F} \quad (3)$$

Thus the ageostrophic component is always directed ‘to the right’ of \mathcal{F} (in the northern hemisphere).

To visualize these balances, consider Fig.1. Lets start with \mathbf{u} : the Coriolis force per unit mass, $-f\hat{\mathbf{z}} \times \mathbf{u}$, must be to the right of the flow, as shown. If the frictional force per unit mass, \mathcal{F} , acts as a ‘drag’ it will be directed opposite to the prevailing flow. The sum of these two forces is depicted by

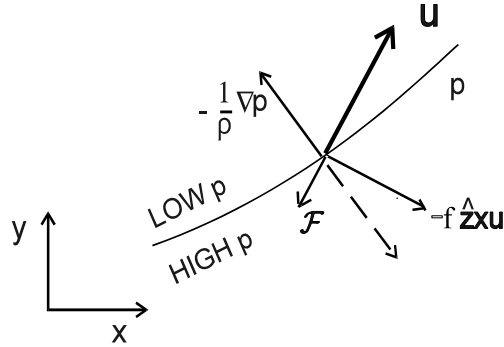


Figure 1: The balance of forces in Eq.(1): the dotted line is the vector sum $\mathcal{F} - f\hat{\mathbf{z}} \times u$ and is balanced by $-\frac{1}{\rho}\nabla p$.

the dashed arrow. This must be balanced by the pressure gradient force per unit mass, as shown. Thus, the pressure gradient is no longer normal to the wind vector, or (to say the same thing) the wind is no longer directed along the isobars. Although there is still a tendency for the flow to have low pressure on its left, there is now a (frictionally-induced) component down the pressure gradient (toward low pressure).

Thus we see that in the presence of \mathcal{F} , the flow speed is subgeostrophic (less than geostrophic) and so the Coriolis force (whose magnitude is proportional to the speed) is not quite sufficient to balance the pressure gradient force. Thus the pressure gradient force ‘wins’, resulting in an ageostrophic component directed from high to low pressure. The flow ‘falls down’ the pressure gradient slightly.

We can readily demonstrate the role of Ekman layers in the laboratory as follows.

1.1 GFD Lab IX - Ekman layers: frictionally-induced cross-isobaric flow

We bring a cylindrical tank filled with water up to solid-body rotation at a speed of 5 rpm, say. A few crystals of Potassium Permanganate are dropped

in to the tank — they fall through the water column and settle on the base of the tank. Note the Taylor Columns. We drop paper dots on the surface to act as tracers of upper level flow and reduce the rotation rate by 10% or so. The fluid continues in solid rotation creating a cyclonic vortex (same sense of rotation as the table) with lower pressure in the center and higher pressure near the rim of the tank. The dots on the surface describe concentric circles and show little tendency to toward radial flow. However at the bottom of the tank we see plumes of dye spiral inward to the center of the tank at about 45° relative to the geostrophic current — see Fig.2, top panel. Now we increase the rotation rate. The relative flow is now anticyclonic with high pressure in the center and low pressure on the rim. Note how the plumes of dye sweep around to point outward — see Fig.2, bottom panel.

In each case we see that the rough bottom of the tank slows the currents down there, and induces cross-isobaric flow from high to low pressure, as schematized in Fig.3. Above the frictional layer — which, as mentioned above, is called the ‘Ekman layer’ — the flow remains close to geostrophic.

1.2 Ageostrophic flow in atmospheric highs and lows

Ageostrophic flow is clearly evident in the bottom kilometer or so of the atmosphere where the frictional drag of the rough underlying surface is directly felt by the flow. For example Fig.4 shows the surface pressure field and wind at the surface at 12GMT on June 21st, 2003. We see that the wind broadly circulates in the sense expected from geostrophy — anticyclonically around highs and cyclonically around the lows. But the surface flow also has a marked component directed down the pressure gradient, ‘filling up’ the lows and ‘emptying out’ the highs. This is due to frictional drag at the ground. The sense of the ageostrophic flow is exactly the same as that seen in laboratory experiment IX — compare with Fig.2 — and sketched schematically in Fig.3.

1.2.1 A simple model of winds in the Ekman layer

Eq.(1) can be solved to give a simple expression for the wind in the Ekman layer. Let us suppose that the x -axis is directed along the isobars and that the surface stress is distributed evenly throughout the depth of the Ekman layer to become small at $z = \delta$, where δ is the depth of the Ekman layer such that



Figure 2: Ekman flow in a low pressure system (top) and a high pressure system (bottom) revealed by permanganate crystals on the bottom of a rotating tank. The black dots are floating on the free surface and mark out circular trajectories around the center of the tank directed anticlockwise (top) and clockwise (bottom).

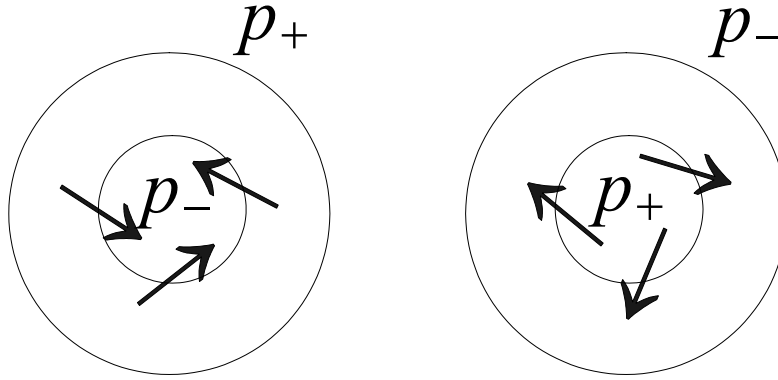


Figure 3: Flow spiralling in to a low and out of a high in a bottom Ekman layer. In both cases the ageostrophic flow is directed from high pressure to low pressure i.e. down the pressure gradient.

$$\mathcal{F} = -\frac{k}{\delta} \mathbf{u} \quad (4)$$

where k is a drag coefficient that depends on the roughness of the underlying surface. Note that the minus sign ensures that \mathcal{F} acts as a drag on the flow. Then the momentum equations become:

$$\begin{aligned} -fv &= -k\frac{u}{\delta} \\ fu + \frac{1}{\rho} \frac{\partial p}{\partial y} &= -k\frac{v}{\delta} \end{aligned} \quad (5)$$

Note that v is entirely ageostrophic, being the component directed across the isobars. But u has both geostrophic and ageostrophic components.

Solving Eq.(5) gives:

$$u = -\frac{1}{\left(1 + \frac{k^2}{f^2\delta^2}\right)} \frac{1}{\rho f} \frac{\partial p}{\partial y}; \quad \frac{v}{u} = \frac{k}{f\delta} \quad (6)$$

Note that the wind speed is *less* than its geostrophic value and if $u > 0$, then $v > 0$.

In typical meteorological conditions, $\delta \sim 1$ km, k is between 1 \rightarrow $1.5 \times 10^{-2} \text{ m s}^{-1}$ and $\frac{k}{f\delta} \sim 0.1$. So the wind speed is only slightly less than

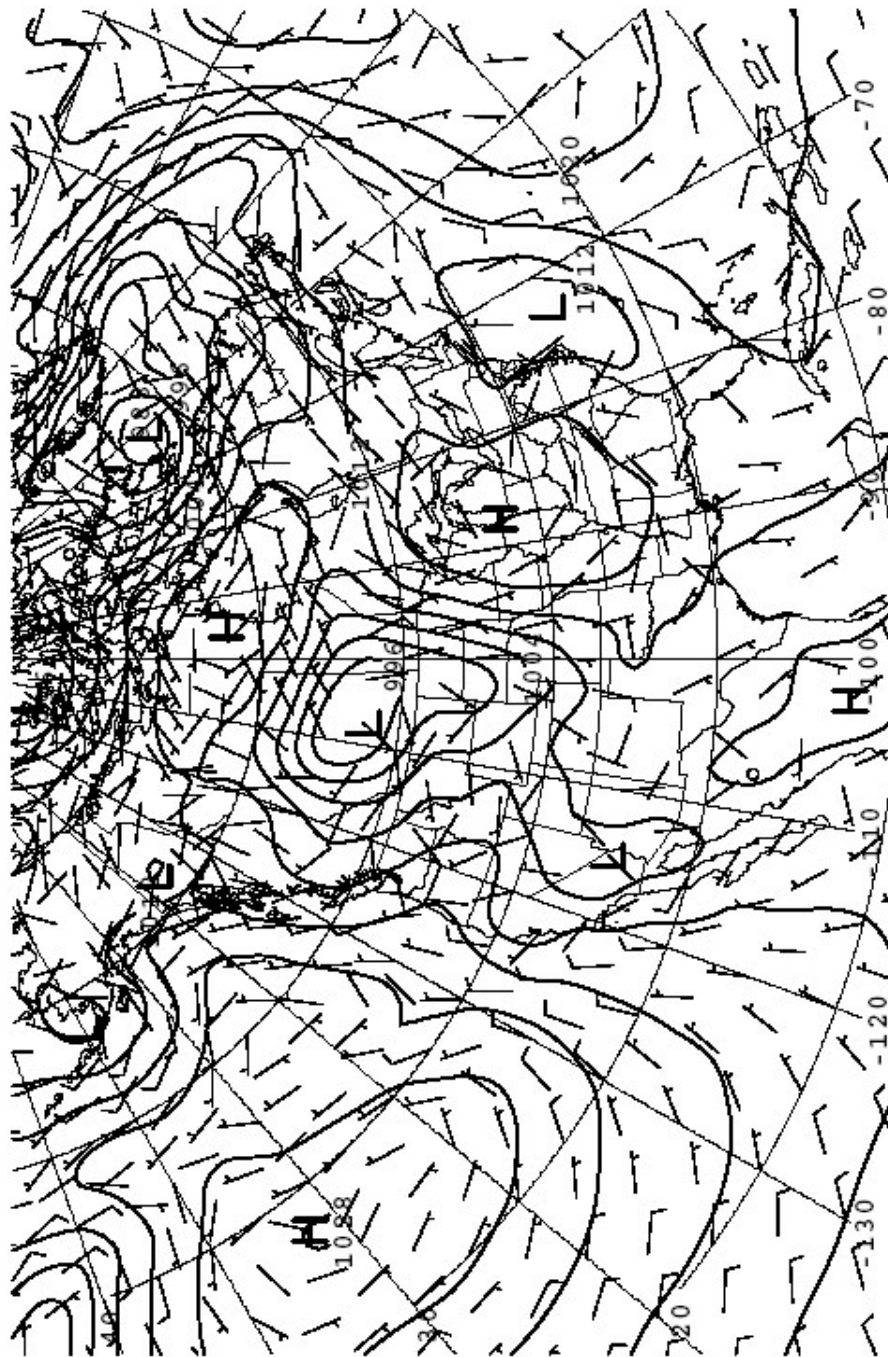


Figure 4: Surface pressure field and wind at the surface at 12GMT on June 21st, 2003. The contour interval is 4 mbar. One full quiver represents a wind of 10 m s^{-1} ; one half quiver a wind of 5 m s^{-1} .

geostrophic, but the wind blows across the isobars at an angle of some 6 to 12°. The cross isobaric flow is strong (k large over land) where the friction layer is shallow (δ small) and in low latitudes (f small). Over the ocean, for example, where k is small, the flow is much closer to its geostrophic value than over land.

Ekman developed a theory of the boundary layer in which he set $\mathcal{F} = \mu \frac{\partial^2 \mathbf{u}}{\partial z^2}$ in Eq.(1) where μ was a constant eddy viscosity. He obtained what are now known as ‘Ekman spirals’ in which the current spirals from its most geostrophic to its most ageostrophic value. But such details depend on the precise nature of \mathcal{F} , which in general is not known. Qualitatively, the most striking and important feature of the Ekman layer solution is that the wind in the boundary layer has a component directed toward lower pressure; this feature is independent of the *details* of the turbulent boundary layer.

1.2.2 Vertical motion induced by Ekman layers

Unlike geostrophic flow, ageostrophic flow is not horizontally non-divergent; on the contrary, its divergence drives vertical motion:

$$\nabla \cdot \mathbf{u}_{ag} + \frac{\partial \omega}{\partial p} = 0$$

This has implications for the behavior of weather systems. Fig.5 shows schematics of a cyclone (low pressure system) and an anticyclone (high pressure system). In the free atmosphere, where the flow is geostrophic, the flow just rotates around the system; cyclonically around the low, anticyclonically around the high. Near the surface in the Ekman layer, however, the wind deviates toward low pressure, inward in the low, outward from the high. Because the horizontal flow is convergent into the low, mass continuity demands a compensating vertical outflow. This *Ekman pumping* produces ascent — and, in consequence, cooling, clouds and possibly rain — in low pressure systems. In the high, the divergence of the Ekman layer flow demands subsidence (through *Ekman suction*) and are dry, which is why high pressure systems tend to be characterized by clear skies.

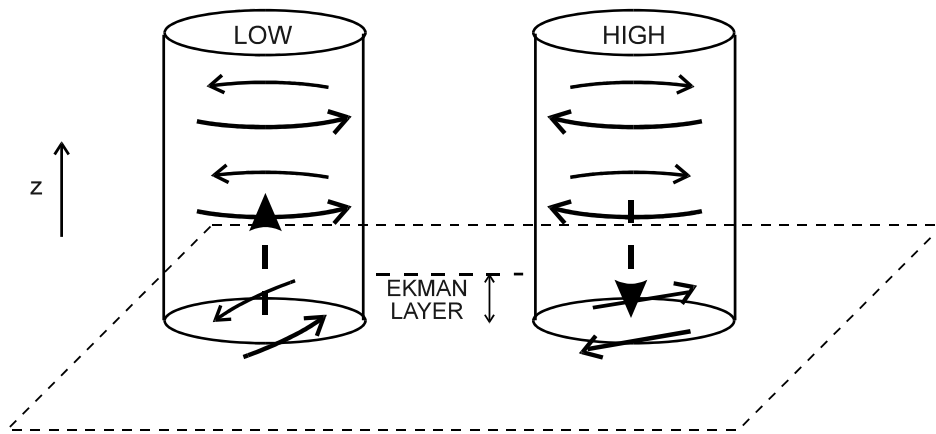


Figure 5: Schematic diagram showing the direction of the frictionally-induced ageostrophic flow in the Ekman layer induced by low pressure and high pressure systems. There is flow in to the low inducing rising motion (the dotted arrow) and flow out of a high inducing sinking motion.