Taylor Columns

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Abstract. In this experiment we explore the effects of rotation on the dynamics of fluids. Using a rapidly rotating tank we demonstrate that an attempt to introduce gradients in velocity in the direction parallel to the axis of rotation in a geostrophically balanced, constant density fluid leads to the formation of Taylor columns. The columns are visualized by tracking flows by adding dye to the fluid. Additionally, the Rossby number of the flow was calculated to be $Ro = 10^{-3}$ in order to verify that the fluid is in geostrophic balance. Next, we prove that geostrophic balance is a good approximation for most flows in the earth's atmosphere by demonstating that the Rossby number of typical flows in the atmosphere is very small and that the geostrophi wind closely matches the observed wind. Finally, we prove that Taylor columns are not observed in the atmosphere because the density is not constant.

1. Introduction. Imagine dragging a small object slowly through a stationary fluid. Our intuition tells us that the motion of the object should disturb the flow in all directions, creating a "wake". The case of pulling an object through a rotating flow might appear to be equivalent, but as is shown in Figure 1-1, if certain conditions are met, the flow created in a rotating fluid is drastically different. The obstacle at the bottom moves slowly through a rotating fluid. Rather than disturbing the flow in all directions, as one would expect, the object pulls a column of fluid parallel to the axis of rotation with it as it moves. Thus, in a rotating environment, a slowly moving object (nearly) immobilizes an entire column of fluid parallel to the rotation axis, behaving nearly as a solid cylinder extended parallel to the rotation axis. This amazing and unintuitive phenomenon was first hypothesized by G.I. Taylor in 1923, and the columns formed are popularly known as Taylor columns.

In this paper we will explore how "geostrophic balance" in an incompressible fluid leads to the condition that there can be no gradients in a direction parallel to the axis of rotation. We show how an attempt to create gradients in the vertical direction by disturbing the flow leads to the formation of Taylor columns. In our tank experiment, we visualize these Taylor columns by adding dye and tracking its movement. Next, we explore geostrophic balance in the atmosphere. We compare observed winds to the geostrophic wind and conclude that geostrophic balance is a very good approximation for the atmosphere. Finally, we explore the reasons why Taylor columns are not observed in the atmosphere although the atmosphere is in geostrophic balance.

2. The Taylor-Proudman Theorem. Taylor predicted the existence of columns of fluid when rotating fluids are disturbed based on a theorem proved by Joseph Proudman in 1915. The theorem, which is now known as the Taylor-Proudman Theorem, states that in a "geostrophically balanced" flow in an incompressible fluid, there can be no gradients in velocity perpendicular to the rotation axis. We will now prove the Taylor-

Figure 1-1: The obstacle at the bottom moves slowly through a rotating fluid. Rather than disturbing the flow in all directions, as one would expect, the object pulls a column of fluid parallel to the axis of rotation with it as it moves.

Proudman Theorem by considering the dominant forces which must balance in the Navier-Stoke's Equations.

The Navier-Stoke's Equations for a fluid of density ρ and kinematic viscosity ν rotating with angular velocity $\mathbf{\Omega} = (0, 0, \omega)$ in an external gravity field $\mathbf{g} = (0, 0, -g)$ are

The Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2-1}$$

The Momentum Equation:

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}}_{\text{interia}} + \underbrace{\mathbf{\Omega} \times (\Omega \times \mathbf{r})}_{\text{centrifugal}} = \underbrace{\frac{-1}{\rho}\nabla P}_{\text{pressure gradients}} + \underbrace{\nu \nabla^2 \mathbf{u}}_{\text{viscous}} - \underbrace{2\mathbf{\Omega} \times \mathbf{u}}_{\text{coriolis}} - \underbrace{\mathbf{g}}_{\text{gravity}} \tag{2-2}$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector and $\mathbf{r}$ is the position vector. The centrifugal and coriolis forces are "fictitious forces" which arise because we are attempting to apply Newton's Law ($\mathbf{F} = m\mathbf{a}$) in a non-inertial (rotating) reference frame. The momentum equation can be simplified by noting that since gravity is a potential field, it can be written as $\mathbf{g} = \nabla \phi$. The centrifugal force term can also be rewritten as a gradient of a potential via the vector identity $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -1/2\nabla(\mathbf{\Omega} \times \mathbf{r})^2$. Therefore, we can combine gravity and the centrifugal force into a single term $\nabla \Phi \equiv \nabla(\phi + 1/2(\mathbf{\Omega} \times \mathbf{r})^2)$, which we consider as a "modified" gravitational force.

The second step in simplifying the equation is to consider the relative importance of the different forces in the types of flows that interest us. We define the characteristic velocity of the system U, the characteristic length scale L, and the characteristic time-scale $\tau = L/U$. Since the Reynolds number of the flow, which is defined as the ratio of inertial forces to viscous forces, is large for all flows on the length scales that we are considering , viscosity can be neglected. Both inertial terms scale as $\frac{U^2}{L}$, and the coriolis term scales as ωU. If the flows are much weaker than the rotation of the system, then the Rossby number

$$Ro \equiv \frac{\text{intertia}}{\text{coriolis}} = \frac{U}{\omega L} \quad \text{(2-3)}$$

is much less than 1. Therefore, the inertial terms can be neglected, and the equations of motion reduce to

$$\frac{-1}{\rho}\nabla P = 2\mathbf{\Omega} \times \mathbf{u} - \nabla\Phi. \quad \text{(2-4)}$$

The vertical component of this equation is termed “hydrostatic balance” and the vertical components are termed “geostrophic balance”. Taking the curl of both sides yields

$$\nabla \times \frac{-1}{\rho}\nabla P = \nabla \times (2\mathbf{\Omega} \times \mathbf{u} - \nabla\Phi).$$

If ρ is constant the left-hand side vanishes because the curl of a gradient is always zero. For constant $\mathbf{\Omega}$, the right-hand side reduces to $2(\mathbf{\Omega}\nabla \cdot \mathbf{u} - (\mathbf{\Omega} \cdot \nabla)\mathbf{u})$. For an incompressible fluid (ρ = constant) the continuity equation reduces to $\nabla \cdot \mathbf{u} = 0$, and therefore the first term on the right hand side vanishes. Therefore,

$$\mathbf{\Omega} \cdot \nabla\mathbf{u} = 0.$$

Since $\mathbf{\Omega} = \omega\hat{\mathbf{z}}$,

$$\mathbf{\Omega} \cdot \nabla\mathbf{u} = \frac{\partial\mathbf{u}}{\partial z} = 0,$$

which implies that the fluid velocity is independent of z. Hence, we have proved that in an incompressible fluid in geostrophic balance, the velocity in the direction parallel to the axis of rotation must be constant.

3. Our Experiment. In our experiment we form Taylor columns by creating a flow over a small object at the bottom of a rotating tank. The experimental setup is shown in Figure 3-1. A 10^o arc is drawn on the bottom of the tank in order to aid us in measuring velocities. A small disk is placed at the bottom of a large cylindrical tank approximately 2/3 of the way from the center of the tank, and the tank is then filled with water. The tank is placed on a rotating table which is equipped with a camera that co-rotates with the table. The table is set to rotate at a relatively high rotation rate (10.52 rpm), and the fluid is allowed to “spin up” until it is undergoing solid body rotation. The mechanism for the spin-up of the fluid is the diffusion of viscosity from the bottom and walls of the tank. In order to ensure that the fluid is undergoing solid body rotation, a small paper dot is placed on the surface and observed in the rotating frame of reference with the camera. If the dot is stationary in the rotating frame, the fluid is undergoing solid body rotation.

Next, the rotation rate of the table is slowed down very slightly. Since the water will continue to move at its initialrotation rate for a finite amount of time, there will be a small relative velocity between the disk sitting on the bottom of the tank and the water in the tank. If we move into the frame of reference of the rotating tank (as observed

Figure 3-1: This figure shows the setup of the experiment. The tank has a radius R=29.6 cm, and the disk was placed 16.9 cm from the center of the tank. The tank is set to rotate about a vertical axis at a speed of 10.52 rpm. The black lines on the bottom of the tank subtends an angle of 10^o and are used in order to estimate the Rossby number for the flow.

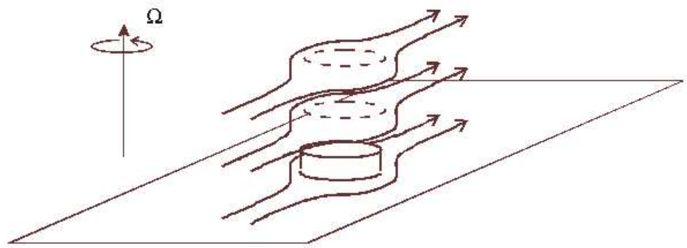

Figure 3-2: This figure illustrates the velocity of the flow relative to the obstacle after the rotation rate of the tank has been slowed slightly. If the relative flow is sufficiently small, the Taylor-Prundam theorem can be applied and Taylor columns will be observed in the flow, as illustrated. Figure taken from 12.307 Notes on the Taylor Column Experiment (2004).

by the camera) the water is seen to slowly flow past the disk, as is illustrated in Figure 3-2.

Before describing the experiment further, I will outline why we expect to observe Taylor columns in our experiment. Since the water is incompressible and the velocity of the relative flow is much smaller than the rotation rate, we can apply the Taylor-Proudman theorem. Therefore, there can be no gradients in velocity in the vertical direction. At the bottom of the tank the flow must go around the obstacle. However, since there can be no velocity gradients in the vertical direction, the flow must also go around the obstacle even above the top of the obstacle, as if the obstacle is extended to the top of the tank. This column of stationary fluid above the obstacle is the Taylor column which we hope to observe.

In order to observe the Taylor columns we inject dye into the fluid at the leading edge of the obstacle, see Figure 3-3. If no Taylor columns are present, we expect the dyed fluid above the top of the obstacle to flow over the obstacle. However, as is observed in Figure 3-3, the dye (blue) goes around the obstacle, as if the obstacle was extended to the top of the tank. We also have placed red dye above the obstacle, and as predicted by the Taylor-Proudman theorem, the dye remains stationary over the top of the obstacle.

4. Measurement of Rossby Number. In order to obtain some quantitative information about our flow, we attempt to measure the Rossby number of the flow. We determine the velocity of the fluid relative to the obstacle by measuring how long it takes for a paper dot on the surface to traverse the arc marked on the bottom of the tank. The angle of the arc is $\theta = \pi/18$, and it takes 3/4 minutes for the dot to travel this angular distance. Therefore, the relative angular velocity $\omega_{\text{rel}} = 2\pi/27$. Since the puck was located at a distance $R = 16.5$ cm from the center of the tank, the relative velocity of the fluid past the puck U is $U = R\omega_{\text{rel}} = 1.2\pi$. The rotation rate of the tank is $\Omega = 10.52rpm = 21.04\pi$ radians/min. Taking the characteristic length scale L to be the radius of the tank, which is 29.6 cm, the Rossby number is

$$Ro = \frac{U}{2\Omega L} = 10^{-3}.$$

Figure 3-3: This figure shows the Taylor columns created in our flow. Blue dye was placed on the leading edge of the obstacle ino order to show that the flow goes around the obstacle, as if the obstacle was extended to the top of the tank. Red dye was placed on the top of the obstacle, and as predicted by the Taylor-Proudman theorem, the dye remains stationary over the top of the obstacle. The red dye at the far side of the tank is leftover from a previous attempt to use dye to visualize the Taylor columns.

Since Taylor columns are only expected for Rossby numbers much less than 1, our Rossby number is consistent with the fact that we observed Taylor columns in our flow.

5. The Real World: Atmospheric Data. Just as the water in our tank, the atmosphere is in geostrophic balance. This can be seen by proving that the typical Rossby number for flows in the atmosphere is small. Since the earth rotates once in 24 hours, the rotation rate is $\Omega = 2\pi/(24 \times 3600) = 7.27 \times 10^-5\text{sec}^{-1}$. If we take the typical length-scale of the flow L to be on the order of the radius of the earth $L = 6.371 \times 10^6$ m/s. Typical wind speeds U are on the order of tens of meters per second. Therefore, the Rossby number $Ro = \frac{U}{\Omega L} \sim 0.1$. Since the Rossby number is significantly less than 1, inertial terms can be ignored and geostrophic balance is a good approximation for the flow. In this section we will calculate the geostrphic wind in the atmosphere at the 500 mbar pressure surface, and compare them to the observed winds to demostrate that the atmosphere is in geostrophic balance. Then, we will explain why Taylor columns are not observed in the atmosphere although it is in geostrophic balance.

6. The Geostrophic Wind. As discussed in Section 2, the equation for geostrophic motion is

$$f\hat{\mathbf{z}} \times \mathbf{u} + \frac{1}{\rho}\nabla P. \tag{6-1}$$

where $f = 2 * \Omega \sin\phi$ is the Corliolis parameter, which quantifies the vertical component of the earth's rotation rate at latitude ϕ, $\mathbf{u}$ is the velocity, P is the pressure, and $\hat{\mathbf{z}}$ is the unit vector in the vertical direction. The velocity defined by the geostrophic flow equation is

$$\mathbf{u_g} = \frac{1}{f\rho}\hat{\mathbf{z}} \times \nabla P \tag{6-2}$$

In component form the equation is

$$\mathbf{u_g} = (u_g, v_g, w_g) = \frac{1}{f\rho}\Big(\frac{-\partial P}{\partial y}, \frac{\partial P}{\partial x}, 0\Big). \tag{6-3}$$

The geostrophic flow is perpendicular to the pressure gradient, and therefore it is along the isobars of the flow. The speed of the geostrophic flow is proportional to the pressure gradient.

However, it is convenient to express the geostrphic wind in terms of pressure coordinates because atmospheric data is generally measured on a constant pressure surface rather than on a constant height surface. The geostrophic wind equation expressed in pressure coordinates is

$$(u_g, v_g) = \Big(-\frac{g}{f}\frac{\partial z}{\partial y}, \frac{g}{f}\frac{\partial z}{\partial x}\Big) \tag{6-4}$$

Figure 5 is a plot of the observed winds at the 500 mbar pressure surface over Japan. The contours show the height of the 500 mbar pressure surface in meters and the vectors show the wind velocity. Each quiver represents a wind speed of 10 m/s and the direction of the wind blows from the side of the vector with the quiver. Figure 6 is a plot of the same area, but now the geostrophic wind (as calulated by Equation (6-4) rather than the observed wind is plotted. As can be seen by comparing Figures 5 and 6, the geostrophic wind is very close to the observed wind, which indicates that geostrophic balance is a good approximation for the atmosphere. The similarity of the geostrophic wind to the observed wind can be made even more explicit by calculating the ageostrophic wind, which is the difference of the observed wind and the geostrophic wind, as is done in Figure 7. The deviations between the observed wind and the geostrophic wind are less than 5 m/s. Therefore, geostrophic balance is a good approximation for the earth's atmosphere.

7. The Atmosphere: A Compressible Fluid. Since the atmosphere is in geostrophic balance, it begs the question of whether Taylor columns are seen in the atmosphere. It might be quite wild to imagine that a light wind blowing over an obstacle might cause a stationary column of air above the obstacle. However, Taylor columns are not seen in the atmosphere. This is due to the fact that the density is not constant in the earth's atmosphere.

In the atmosphere density is a function of pressure and temperature, $\rho = \rho(P, T)$. Due to the density variations with temperature, an expression comparable to the Taylor-Proudman Theorem is untidy when height is used as the vertical coordinate. The equations are much simpler if we use pressure as our vertical coordinate. Pressure is also a convenient coordinate for atmospheric observations because atmospheric data is generally measured on a constant pressure surface rather than on a constant height surface.

We saw that geostrophic balance holds in the atmosphere both by calculating the Rossby number of typical flows in the atmosphere We saw that geostrophic balance holds in the atmosphere both by calculating the Rossby number of typical flows in the atmosphere In pressure coordinates, the hydrostatic relation can be expressed as

$$\frac{\partial z}{\partial P} = -\frac{1}{g\rho} \tag{7-1}$$

As seen in section (6), the geostrophic wind can also be expressed in pressure coordinates. Taking the P-derivative of the x-component of Equation (6-4) and using the hydrostatic relation, yields

$$\frac{\partial u}{\partial P} = \frac{1}{f}\frac{\partial}{\partial y}\Big(\frac{1}{\rho}\Big)_P$$

Approximating the atmosphere as an ideal gas, we use the equation of state $1/\rho = RT/P$ where T is the temperature and $R = 287Jkg^{-1}K^{-1}$ is the is the gas constant for dry air. Substituting the equation of state into the above equation yields

$$\frac{\partial u}{\partial P} = \frac{R}{fP}\Big(\frac{\partial T}{\partial y}\Big)_P \tag{7-2}$$

Similarly, for v we find

$$\frac{\partial v}{\partial P} = -\frac{R}{fP}\Big(\frac{\partial T}{\partial x}\Big)_P \tag{7-3}$$

Equations (6-3) and (6-4) are the " thermal wind equations" in pressure coordinates. They the analogue of the Taylor-Proudman Theorefor a fluid where density is not constant. We can see that horizontal variations in temperature cause vertical gradients in velocity.

8. Conclusions. We proved the Taylor-Proundman Theorem: for a geostrophically balanced flow in a rotating fluid with a constant density, there can be no gradients in the direction parallel to the axis of rotation. We demonstrated in the laboratory that an attempt to introduce gradients in velocity in the vertical direction leads to the formation of Taylor columns. Finally, we proved that geostrophic balance is a good approximation for most flows in the earth's atmosphere. However, Taylor columns are not observed in the atmosphere because the density is not constant.

9. Further Work. One aspect of the experiment that could be improved significantly is the method of visualization of the Taylor columns. Using a high molecular weight dye, such as phenol blue, would significantly reduce diffusive flows and make visualization easier. An interesting extension to the experiment might be to try the experiment with a fluid in which the density is not constant, such as in a fluid with a gradient in salt concentration.

References

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12.307 Notes on the Taylor Column Experiment (2004)
12.307 Project 1 Notes (2004)